

NOTE ON THE EULER EQUATIONS IN C^k SPACES

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ABSTRACT. In this note, using the ideas from our recent article [2], we prove strong ill-posedness for the 2D Euler equations in C^k spaces. This note provides a significantly shorter proof of many of the main results in [1]. In the case $k > 1$ we show the existence of initial data for which the k th derivative of the velocity field develops a logarithmic singularity immediately. The strong ill-posedness covers $C^{k-1,1}$ spaces as well. The ill-posedness comes from the pressure term in the Euler equation. We formulate the equation for $D^k u$ as:

$$\partial_t D^k u = D^{k+1} p + l.o.t.$$

and then use the non-locality of the map $u \rightarrow p$ to get the ill-posedness. The real difficulty comes in how to deal with the "l.o.t." terms which can be handled by special commutator estimates.

1. INTRODUCTION

In this note we give a short proof of the strong ill-posedness of the Euler equations for incompressible flow in C^k spaces. Our proof works in the whole space case as well as the case of periodic boundary conditions and the bounded domain case. We use the non-locality of the map $u \rightarrow p$ to get growth in C^k spaces (since Riesz transforms are unbounded on these spaces!). The proof uses classical commutator estimates which estimate the commutation of a Riesz transform (or a Calderón Zygmund operator) and composition by a bi-Lipschitz map.

We prove the following theorem.

Theorem 1.1. *The Euler equations are strongly ill-posed in C^k spaces for $k \geq 1$. In other words, for every $\epsilon > 0$ there exists initial data $u_0 \in C^k$ such that the unique solution, $u(t)$, of the Euler equations with initial data u_0 leaves C^k immediately.*

We note that very recently Bourgain and Li have proven the same result as above [1]. In their paper the authors cite our work [2] and note that it did not include ill-posedness C^k spaces for $k > 1$. In this note we show that the work in [2] can easily be extended to the case $k > 1$. Moreover we make clear the fact that strong ill-posedness in C^k can be proven quite easily only using commutator estimates without having to rely upon very intricate constructions.

2. THE PROOF

Proof. Note that it suffices to consider the two dimensional Euler equations. Now consider the equation for $D^k u$ which means k spatial derivatives of u , which we take to be a vector of many components.

$\nabla D^{k-1}u$ satisfies the following equation:

$$\partial_t \nabla D^{k-1}u + u \cdot \nabla D^k u + \sum_{j,l}^k Q(D^j u, D^l u) + D^{k-1} D^2 p = 0$$

Recall that

$$\Delta p = \det(\nabla u)$$

so that

$$(D^2 p)_{ij} = (R_i R_j \det(\nabla u))_{ij}$$

We rely upon the following commutator estimate in L^p :

Lemma 2.1. *Let Φ be a bi-Lipschitz measure preserving map. Let $K = \max\{|\Phi - Id|_{Lip}, |\Phi^{-1} - Id|_{Lip}\}$. Let R be a composition of Riesz transforms. Define the following commutator:*

$$[R, \Phi]\omega = R(\omega \circ \Phi) - R(\omega) \circ \Phi,$$

for $\omega \in L^p$.

Then,

$$\|[R, \Phi]\|_{L^p \rightarrow L^p} \leq c_p K.$$

Moreover, $c_p \leq cp$ as $p \rightarrow \infty$.

Now we recall the Lagrangian flow

$$\begin{aligned} \dot{\Phi}(x, t) &= u(\Phi(t, x), t) \\ \Phi(x, 0) &= x. \end{aligned}$$

Because u is divergence free, Φ is measure preserving. Furthermore, $\Phi(x, -t) = \Phi^{-1}(x, t)$.

Now, we may write

$$\Phi(x, t) = x + \int_0^t u(\Phi(x, \tau), \tau) d\tau.$$

Thus,

$$\Phi(\cdot, t) - I = \int_0^t u(\Phi(\cdot, \tau), \tau) d\tau.$$

Consequently,

$$|\Phi - I|_{Lip} \leq t|u|_{Lip}|\Phi|_{Lip}$$

and similarly for $\Phi^{-1}(\cdot, t) = \Phi(\cdot, -t)$.

Furthermore, by Gronwall's lemma,

$$|\Phi|_{Lip} \leq \exp(t|u|_{Lip}).$$

In particular,

$$(2.1) \quad |\Phi - I|_{Lip} \leq t|u|_{Lip} \exp(t|u|_{Lip}).$$

In particular, if u is Lipschitz then the Lagrangian flow-map is controlled. Now, assume that u remains Lipschitz (which in the case $k > 0$ is trivial).

Then we have:

$$\partial_t(D^k u \circ \Phi) + \sum_{j,l}^k Q(D^j u, D^l u) \circ \Phi + \left(R_i R_j D^{k-1} \det(\nabla u) \right) \circ \Phi = 0$$

Now assume that the solution remains in C^k .

Now call $R_i R_j := R$ for short.

$$\partial_t(D^k u \circ \Phi) + \sum_{j,l}^k Q(D^j u, D^l u) \circ \Phi + \left(R D^{k-1} \det(\nabla u) \right) \circ \Phi = 0$$

$$\partial_t(D^k u \circ \Phi) + \sum_{j,l}^k Q(D^j u, D^l u) \circ \Phi + R \left(D^{k-1} \det(\nabla u) \circ \Phi \right) = [R, \Phi] D^{k-1} \det(\nabla u)$$

We are going to take a special initial data $u_0 \in C^k$. Now, suppose that the solution remains in C^k . Note that we can always solve the 2D Euler equations in $W^{k,p}$ and that the solution will be unique when $k > 1$ (in the case $k=1$ we will also get a unique solution the class of velocity fields with bounded curl). Assume that the solution stays in C^k with $|u|_{C^k} \leq M$.

Then we can solve the Euler equations formally using the Duhamel formula:

$$|D^k u \circ \Phi|_{L^p} \geq |D^k u_0 + t R(D^{k-1} \det(\nabla u_0))|_{L^p} - \left| \int_0^t e^{R(t-s)} [R, \Phi] D^{k-1} \det(\nabla u) - Q(D^j u, D^l u) \circ \Phi ds \right|_{L^p}$$

Now suppose that we construct compactly supported initial data u_0 such that $|R D^{k-1} \det(\nabla u_0)|_{L^p} \geq cp$ as $p \rightarrow \infty$.

Then we see that

$$|D^k u \circ \Phi|_{L^p} \geq tcp - C - t(1+tp) |[R, \Phi]|_{L^p \rightarrow L^p} C(M) + tC(M).$$

This is because $\|R\|_{L^p \rightarrow L^p} \lesssim p$ as $p \rightarrow \infty$. Now using the commutator estimate in Lemma 2.1 we get:

$$|D^k u \circ \Phi|_{L^p} \geq tcp - C - t(1+tp) tp \tilde{C}(M) + tC(M)$$

Now take t very small depending upon M then we get:

$$|D^k u \circ \Phi|_{L^p} \geq t\tilde{c}p$$

for t small and all p large.

This contradicts the fact that $|D^k u|_{L^\infty}$ remains bounded.

□

Now, of course we relied upon the existence of an initial data u_0 such that

$$|R D^{k-1} \det(\nabla u_0)|_{L^p} \geq cp.$$

Constructing such initial data is not too difficult. We will copy the construction from our work [2] below in the case $k = 1$. The higher order cases are similar.

We are interested in showing that for some i, j and for some divergence free u , with $\nabla u \in L^\infty$, $D^2 p = R_i R_j \det(\nabla u)$ has a logarithmic singularity. Once that is shown, Lemma 8.2 will follow by a regularization argument.

Take a harmonic polynomial, Q , which is homogeneous of degree 4. In the two-dimensional case, we can take

$$Q(x, y) := x^4 + y^4 - 6x^2 y^2, \\ \Delta P = 0.$$

Define

$$G(x, y) := Q(x, y) \text{Log}(x^2 + y^2).$$

Notice that

$$(2.2) \quad \partial_i \partial_j \Delta G \in L^\infty(B_1(0)), i, j \in \{1, 2\}.$$

Notice, on the other hand, that

$$(2.3) \quad \partial_{xxyy} G = -24 \text{Log}(x^2 + y^2) + H(x, y),$$

with $H \in L^\infty(B_1(0))$. In particular, $\partial_{xxyy} G$ has a logarithmic singularity at the origin—and the same can be said about $\partial_{xxxx} G$ and $\partial_{yyyy} G$.

Define $\tilde{u} = \nabla^\perp \Delta G$. Then, by 8.12, $\nabla \tilde{u} \in L^\infty(B_1(0))$. Moreover, by definition,

$$R_i R_j \nabla \tilde{u} = \nabla \nabla^\perp \partial_{ij} G.$$

Thus, for example, $R_1 R_2 \nabla \tilde{u}_{1x} = \partial_{xxyy} G$ has a logarithmic singularity in $B_1(0)$. Unfortunately, we are interested in showing that $R_i R_j \det(\nabla u)$ has a logarithmic singularity for some i, j , not $R_i R_j \nabla u$. To rectify this, we choose

$$u = \delta \nabla^\perp \Delta(\chi G) + \eta(2y\chi + y^2 \partial_y \chi, y \partial_x \chi),$$

where η, δ are small parameters which will be determined and χ is a smooth cut-off function with:

$$\chi = 1 \text{ on } B_1(0), \\ \chi = 0 \text{ on } B_2(0)^c.$$

Note that u is divergence free and

$$u \equiv \delta \nabla^\perp \Delta G + \eta(y, 0) \text{ on } B_1(0).$$

Therefore,

$$\nabla u = \delta \begin{bmatrix} -\partial_{xy} \Delta G & -\partial_{yy} \Delta G \\ \partial_{xx} \Delta G & \partial_{xy} \Delta G \end{bmatrix} + \eta \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

In particular,

$$\det(\nabla u) = \eta \delta \partial_{xx} \Delta G + \delta^2 J(x, y),$$

where J is a bounded on $B_1(0)$.

Now consider $R_2 R_2 \det(\nabla u)$:

$$R_2 R_2 \det(\nabla u) = \eta \delta \partial_{xxyy} G + \delta^2 R_2 R_2 J.$$

Now, by (8.13), we have

$$R_2 R_2 \det(\nabla u) = \eta \delta (-24 \operatorname{Log}(x^2 + y^2) + H(x, y)) + \delta^2 R_2 R_2 J,$$

with H and J bounded. Now, recall that $R_2 R_2$ maps L^∞ to BMO and that any BMO function can have at most a logarithmic singularity.

Thus,

$$|R_2 R_2 \det(\nabla u)| \geq 24\eta \delta \operatorname{Log}(x^2 + y^2) - C\delta^2 \operatorname{Log}(x^2 + y^2) - |H(x, y)|.$$

Choose $\delta \ll \eta$ and we see that, near $(0, 0)$

$$|R_2 R_2 \det(\nabla u)| \geq 12\delta^2 \operatorname{Log}(x^2 + y^2).$$

Taking $\delta \leq C$ small enough, we see that $|\nabla u| \leq 1$ but $|R_2 R_2 \det(\nabla u)| \geq c \operatorname{Log}(x^2 + y^2)$, for some small c .

This completes the construction.

REFERENCES

- [1] J. Bourgain and D. Li. Strong illposedness of the incompressible Euler equation in integer C^m spaces *preprint*, *arXiv:1405.2847*.
- [2] T.M. Elgindi and N. Masmoudi L^∞ ill-posedness for a class of equations arising in hydrodynamics. *preprint*, *arXiv:1405.2478*.